Towards Weyl fermions on the lattice without artefacts

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Abstract: In spite of the breakthrough in non-perturbative chiral gauge theories during the last decade, the present formulation has stubborn artefacts. Independently of the fermion representation one is confronted with unwanted CP violation and infinitely many undetermined weight factors. Renormalization group identifies the culprit. We demonstrate the procedure on Weyl fermions in a real representation.

Keywords: Lattice Quantum Field Theory, Renormalization Group.

Contents

1. Introduction and summary

The non-perturbative formulation of chiral gauge theories has been blocked for a long time by the problem of chiral symmetry in vector theories. Although the Ginsparg-Wilson (GW) relation [1], which is coding chiral symmetry on the lattice, was around since 1982, it took a long time to appreciate its significance [2], find acceptable Dirac operators satisfying the GW relation [3] and identify the modified chiral symmetry transformation [4]. Rather different approaches [5] converged to the conclusion that chiral vector theories (like QCD with zero quark masses) can be defined non-perturbatively without compromising any of the basic principles of a QFT. This development lead to a formulation of chiral gauge theories on the lattice where it became possible to demonstrate that the theory has exact gauge symmetry and satisfies the basic requirements of a QFT in every order in a perturbative expansion, or non-perturbatively [6].

The starting point in the works [6] is a vector gauge theory where the Dirac operator satisfies the GW relation. The fermions are chosen to be in a gauge anomaly free representation of the target chiral gauge theory. The next step is to introduce left-handed fermion fields. It turns out that under the condition of locality the gauge field dependent projectors [7] show an asymmetry between fermions and anti-fermions. This is the source of fermion number violation in the chiral theory, and so, it is a welcome feature. On the other hand, this asymmetry breaks CP at the same time [8]. The rôle of this unwanted symmetry breaking in the chiral theory is not completely understood. Since the generated θ-term is a 4-dimensional operator, this CP violation creates a tuning problem. The number of degrees of freedom depends on the topological sector which, on one hand creates fermion number violation (as expected), on the other hand, is responsible for the fact that the chiral theory falls into topological sectors, where the relative weights between these sectors remain undetermined.

For the problem discussed here, the basic step in the process above is the first one: choosing the Dirac operator in the vector theory. Using renormalization group (RG) language, which was the way to find the GW relation in the first place [1], we might define

the lattice action in the Gaussian fixed point [9]. For example, one might use a simple "blocking out of continuum" step [10]. We shall consider symmetries like the axial and vector flavour transformations, where the gauge field is not influenced. Concerning the form of the block transformation one can make the following two statements:

1. One *can* break any of such global symmetries in the block transformation if this is convenient.

2. One *must* break in the block transformation those symmetries which are anomalous in the quantized target theory.

The first statement follows from the fact that the RG group transformation in ref. [9] does not change the physical content of the symmetries considered. The symmetries of the action broken by the blocking do not disappear, rather the symmetry transformations have a new form [4]. Actually, there exists a simple and general procedure to find these modified symmetry transformations [11]. The modified symmetry transformations might be gauge field dependent and the integration measure of the path integral is then not necessarily invariant. The symmetry is called anomalous if the measure cannot be kept invariant without violating basic principles of a QFT.

The second statement is obvious: if a symmetry of the formal continuum action is not broken by the blocking which brings it to the lattice, then this symmetry and the continuum symmetry transformation will be inherited by the lattice action and the measure. There will be no anomaly and the punishment is non-locality, or unwanted particle content ("doublers"), or something else. In our problem one might introduce fermion number violation by hand in the projectors (rather than via RG), but this step leads to the artefacts discussed before.

We shall demonstrate on a SU(2) gauge theory with two flavours that the RG indeed avoids the artefacts if statement 2. above is respected. The lattice action of the vector theory is invariant under the U(1) axial and U(1) vector flavour transformations which are gauge field dependent. The correct axial anomaly is reproduced by the measure, while the fermion number conservation remains intact in this vector theory. The vector theory falls into left- and right-handed pieces with the gauge field independent projectors $P_{R/L} = \frac{1}{2}(1 \pm \gamma_5)$. The chiral theory has the expected U(1) flavour anomaly breaking fermion number conservation.

This chiral gauge theory with Weyl fermions has been discussed earlier by Suzuki [12] using the standard setup. In spite of the real representation, the artefacts were present there the same way as in theories in complex representations [12].

The difficulty to bring these ideas over to complex representations lies in finding a block transformation which is theoretically obviously correct and technically feasable at the same time.

2. The lattice action with two GW type of relations

We start with a formal vector SU(2) gauge theory with two flavours in the continuum.

With a blocking out of continuum RG step[1] we bring the theory to the lattice. Since we consider a block transformation with fermion number breaking (see statement 2.), it is useful to introduce an 8-component notation in the form[2]

$$\phi(x) = \begin{pmatrix} \psi(x) \\ \overline{\psi}^T(x) \end{pmatrix}. \tag{2.1}$$

In eq. (2.1) the flavour ($i = 1, 2$), colour ($a = 1, 2$) and the Dirac ($\alpha = 1, \ldots, 4$) indices are not shown explicitly. The fixed-point (FP) lattice fermion action is obtained by a simple minimization ([2, 13, 14] and references therein)

$$\frac{1}{2}\varphi^T \mathcal{D}\varphi = \min_{\phi} \left\{ \frac{1}{2}\phi^T D\phi + \frac{1}{2}(\varphi - \Omega\phi)^T \mathcal{E}(\varphi - \Omega\phi) \right\}. \tag{2.2}$$

This equation defines the lattice fermion action (lhs. of eq. (2.2)) for any given fermion configuration φ on the lattice by the rhs. of eq. (2.2), where $\phi_{\min} = \phi_{\min}(\varphi)$. In our notation

$$\varphi_n = \begin{pmatrix} \chi_n \\ \overline{\chi}_n^T \end{pmatrix}, \tag{2.3}$$

while D and Ω are the continuum Dirac operator and the averaging function (including the form of parallel transporting), respectively. The continuum Dirac operator can be written as

$$D = \begin{pmatrix} 0 & -d^T \\ d & 0 \end{pmatrix}, \quad d = \gamma_\mu D_\mu, \tag{2.4}$$

where D_μ is the covariant derivative and d is diagonal in flavour space[3]. The gauge field (U) dependent averaging function has the form

$$\Omega(U) = \begin{pmatrix} \omega(U) & 0 \\ 0 & \omega^*(U) \end{pmatrix}, \tag{2.5}$$

while the matrix $\mathcal{E}$ is chosen to break the U(1) chiral and U(1) fermion number symmetries[4]

$$\mathcal{E} = i\epsilon^c \cdot \epsilon^{fl} \cdot \overline{C} \cdot I, \quad \overline{C} = -\overline{C}^T = -\overline{C}^\dagger, \tag{2.6}$$

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[1]Since finding the Gaussian fixed point is a classical field theory problem [2, 9], the "blocking out of continuum" is equivalent to blocking the Wilson action on a lattice with lattice unit $a \to 0$ assuming that the averagings are matched.

[2]We are indebted to Ferenc Niedermayer for suggesting this formalism.

[3]The path integral over the field ϕ with the action $\frac{1}{2}\phi^T D\phi$ gives the Pfaffian of D, $\mathrm{Pf}(D)$. Here $\mathrm{Pf}(D)^2 = \det D = (\det d)^2$, demonstrating that there is no double counting in the 8-component formulation.

[4]In our conventions $\overline{C} = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$.

where ϵ^{c} and ϵ^{fl} are the 2×2 anti-symmetric ϵ-tensors in colour and flavour space respectively. The matrix I is the 8×8 unit matrix in the notation of eq. (2.1). Due to the identity

$$\epsilon_{\xi\xi'} V^{\xi\eta} V^{\xi'\eta'} = \det V \cdot \epsilon_{\eta\eta'} = \epsilon_{\eta\eta'} \tag{2.7}$$

for $V \in \mathrm{SU}(2)$, we find that the block transformation in eq. (2.2) is gauge invariant and preserves the lattice rotation symmetries. It is also C, P, T and SU(2) flavour and SU(2) chiral invariant. On the other hand, the block transformation breaks U(1) flavour (fermion number) and U(1) chiral symmetries.

For any given lattice configuration φ the corresponding minimizing continuum field reads

$$\phi_{\min}(\varphi) = A^{-1}\Omega^T \mathcal{E}\varphi, \quad A = D + \Omega^T \mathcal{E}\Omega \tag{2.8}$$

which gives for the lattice Dirac operator

$$\mathcal{D} = \mathcal{E} - \mathcal{E}\Omega A^{-1}\Omega^T \mathcal{E}, \tag{2.9a}$$

or, if $\mathcal{D}^{-1}$ exists,

$$\mathcal{D}^{-1} = \Omega D^{-1}\Omega^T + \mathcal{E}^{-1}. \tag{2.9b}$$

In our 8-component notation the matrices related to flavour U(1) axial and vector transformation in the continuum have the form

$$\Gamma_5 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & \gamma_5 \end{pmatrix}, \quad \Gamma_V = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{2.10}$$

These matrices anti-commute with the continuum Dirac operator D (eq. (2.4)) expressing U(1) axial and vector (fermion number) symmetries. The corresponding modified lattice symmetries of the action are coded in two Ginsparg-Wilson type of relations

$$\{\Gamma, \mathcal{E}^{-1}\mathcal{D}\} = 2(\mathcal{E}^{-1}\mathcal{D})\Gamma(\mathcal{E}^{-1}\mathcal{D}), \tag{2.11a}$$

where $\Gamma = \Gamma_5$, or $\Gamma = \Gamma_V$. If $\mathcal{D}^{-1}$ is defined (no zero modes) then eq. (2.11a) can be written as

$$\{\Gamma, \mathcal{D}^{-1}\} = 2\Gamma\mathcal{E}^{-1}, \quad \Gamma = \Gamma_5 \text{ or } \Gamma_V. \tag{2.11b}$$

The modified infinitesimal lattice symmetry transformations read [11]

$$\delta\varphi = i\eta\Gamma(1 - \mathcal{E}^{-1}\mathcal{D})\varphi, \quad \Gamma = \Gamma_5, \text{ or } \Gamma_V. \tag{2.12}$$

With the help of the relations eq. (2.11a) it is easy to show that the lattice action $\frac{1}{2}\varphi^T \mathcal{D}\varphi$ is invariant under the transformations in eq. (2.12).

Switching off the gauge interaction, the Dirac operator $\mathcal{D}$ on the lattice can be constructed explicitly and locality, and the absence of doublers can be confirmed [14].

3. The anomaly in the vector theory

We show now that the measure of the vector theory

$$\mathrm{D}\varphi \equiv \prod_{n,a,i,\alpha} \mathrm{d}\varphi_a(n)_i^\alpha \tag{3.1}$$

is invariant under $\mathrm{U}(1)$ flavour and anomalous under $\mathrm{U}(1)$ axial transformation, as expected[5].

The change of the measure under the $\mathrm{U}(1)$ transformations in eq. (2.12) reads

$$\mathrm{D}\varphi \to \left(1 - i\eta \mathrm{Tr}\left(\Gamma \mathcal{E}^{-1}\mathcal{D}\right)\right)\mathrm{D}\varphi, \quad \Gamma = \Gamma_5, \Gamma_V \tag{3.2}$$

where we used $\mathrm{Tr}\,\Gamma = 0$. We introduce an orthonormal basis to calculate the trace above. The matrices D, $\mathcal{E}$ and $\mathcal{D}$ are $\slashed{V}_5$-hermitian: $D^\dagger = \slashed{V}_5 D \slashed{V}_5, \ldots$, where

$$\slashed{V}_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}. \tag{3.3}$$

We consider the basis spanned by the eigenvectors u_l of the hermitian matrix $\hat{\mathcal{D}} = \slashed{V}_5\mathcal{D}$. The subspace of zero modes of $\hat{\mathcal{D}}$ is the same as that of $\mathcal{D}$. Since $\mathcal{D}^{-1}$ is well defined on the non-zero modes, we can write

$$-\mathrm{Tr}\left(\Gamma\mathcal{E}^{-1}\mathcal{D}\right) = -\sum_{\lambda_l \neq 0}\langle u_l, \Gamma\mathcal{E}^{-1}\mathcal{D}u_l\rangle = -\frac{1}{2}\sum_{\lambda_l \neq 0}\langle u_l, (\Gamma\mathcal{D}^{-1} + \mathcal{D}^{-1}\Gamma)\mathcal{D}u_l\rangle$$

$$= \begin{cases} 0 & \text{if } \Gamma = \Gamma_V \\ -\displaystyle\sum_{\lambda_l \neq 0}\langle u_l, \Gamma_5 u_l\rangle = \sum_{\lambda_l = 0}\langle u_l, \Gamma_5 u_l\rangle, & \text{if } \Gamma = \Gamma_5 \end{cases} \tag{3.4}$$

where we used eq. (2.11b) and the relations

$$\{\Gamma_V, \slashed{V}_5\} = 0, \quad [\Gamma_5, \slashed{V}_5] = 0. \tag{3.5}$$

As eq. (3.4) shows, the $\mathrm{U}(1)$ vector symmetry is respected by the measure, while only the zero modes contribute to the axial anomaly.

From the GW relations in eq. (2.11a) follows that if $\hat{\mathcal{D}}u_0 = 0$, then u_0, $\frac{1}{2}(I \pm \Gamma)u_0$ ($\Gamma = \Gamma_5, \Gamma_V$) and $\slashed{V}_5 u_0^*$ are zero modes of $\mathcal{D}$. We can define then a convenient basis in the space of the zero modes as

$$u_i = \begin{pmatrix} v_i \\ 0 \end{pmatrix}, \quad \bar{u}_i = \begin{pmatrix} 0 \\ v_i^* \end{pmatrix} \quad i = 1, \ldots, N. \tag{3.6}$$

Here v_i and v_i^* can be taken left- or right-handed, $v_{Li} = \frac{1-\gamma_5}{2}v_i$, $v_{Ri} = \frac{1+\gamma_5}{2}v_i$, $v_{Li}^* = \frac{1+\gamma_5}{2}v_i^*$, $v_{Ri}^* = \frac{1-\gamma_5}{2}v_i^*$. We have then N_L, N_R, $\bar{N}_L$, and $\bar{N}_R$ zero modes where

$$\bar{N}_R = N_L, \quad \bar{N}_L = N_R, \quad N_R + N_L = N. \tag{3.7}$$

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[5]In eq. (3.1) the indices a, i, α refer to colour, flavour and Dirac space, respectively.

We obtain for the measure contribution

$$-\mathrm{Tr}\,\left(\Gamma_5 \mathcal{E}^{-1}\mathcal{D}\right) = \sum_i \left(\langle u_i,\, \Gamma_5 u_i\rangle + \langle \bar{u}_i,\, \Gamma_5 \bar{u}_i\rangle\right)$$

$$= N_R + \bar{N}_L - N_L - \bar{N}_R = 2(N_R - N_L). \tag{3.8}$$

The result $\mathrm{i}\eta \cdot 2(N_R - N_L)$ for the change of the measure under an infinitesimal $\mathrm{U}(1)$ axial transformation is independent of the details of the GW relation and is reproduced correctly in eq. (3.8).

4. Fermion number anomaly in the chiral theory

Since both the continuum action and the block transformation in eq. (2.2) fall in left- and right-handed pieces with the standard $\frac{1 \pm \gamma_5}{2}$ projectors, so does the lattice action. In the 8-component notation the chiral field is written as

$$\phi_L = \begin{pmatrix} \psi_L \\ \overline{\psi}_L^T \end{pmatrix} = \mathcal{P}_L \phi. \tag{4.1}$$

The infinitesimal fermion number transformation in the continuum reads

$$\delta\phi_L = \mathrm{i}\eta\Gamma_V \phi_L = -\mathrm{i}\eta\Gamma_5 \phi_L. \tag{4.2}$$

The corresponding transformation on the lattice has the form

$$\delta\varphi_L = -\mathrm{i}\eta\Gamma_5(1 - \mathcal{E}^{-1}\mathcal{D})\mathcal{P}_L \varphi_L \tag{4.3}$$

and the measure is changed by

$$-\mathrm{Tr}_L\,\Gamma_5(1 - \mathcal{E}^{-1}\mathcal{D}). \tag{4.4}$$

The trace in eq. (4.4) is taken in the left-handed zero mode space only. One obtains from eq. (3.8)

$$-\mathrm{Tr}_L\,\Gamma_5(1 - \mathcal{E}^{-1}\mathcal{D}) = \bar{N}_L - N_L \tag{4.5}$$

which shows that the fermion number is anomalous in the chiral theory.

Acknowledgments

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